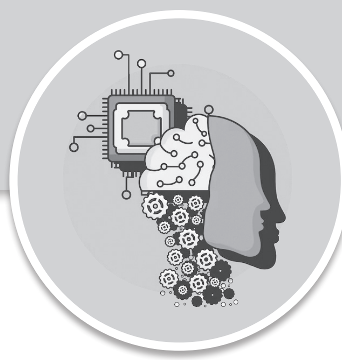
DATA SCIENCE & ARTIFICIAL INTELLIGENCE

Calculus and Optimization



Comprehensive Theory
with Solved Examples *and* Practice Questions

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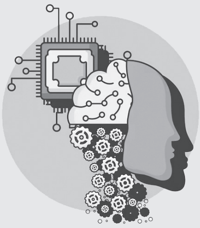

EDITIONS

First Edition : 2025

Second Edition : 2026

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Calculus and Optimization

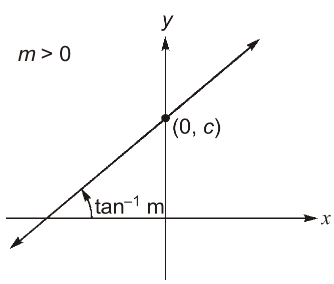
CHAPTER

1

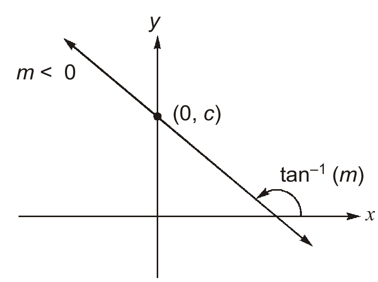
Limit, Continuity and Differentiability

1.1 GRAPHS OF BASIC FUNCTIONS

1.



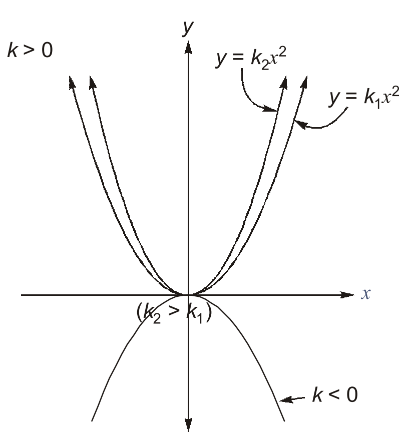
(a)



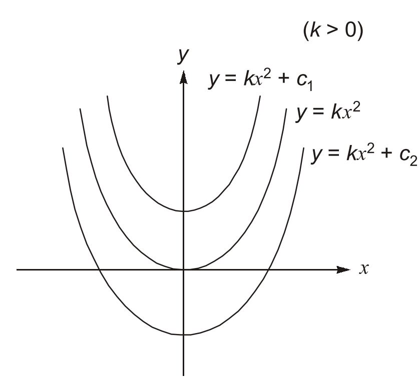
(b)

$m \rightarrow$ slope i.e., angle that curve makes with x-axis in anticlockwise direction.
$c \rightarrow$ intercept i.e., value of y for $x = 0$.

2. $y = kx^2$
(similar shape of kx^4, kx^6 ....)



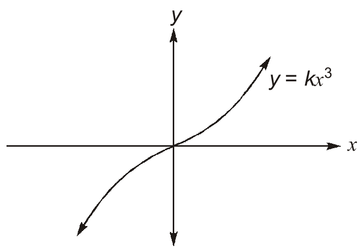
3. $y = kx^2 + c$



$c_1 > 0$; $c_2 < 0$

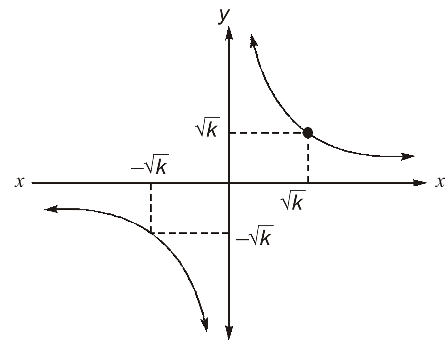
4. $y = kx^3$

(similar shape for kx^5, kx^7 ....)



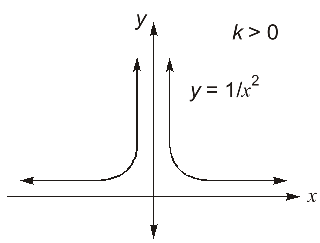
5. $y = \frac{k}{x}$ or $xy = k$

(similar shape for $\frac{k}{x^3}, \frac{k}{x^5}$ ...)



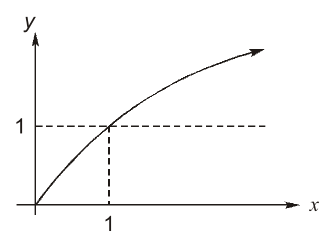
6. $y = \frac{k}{x^2}$

(similar shape for $\frac{k}{x^4}, \frac{k}{x^6}$ .....)

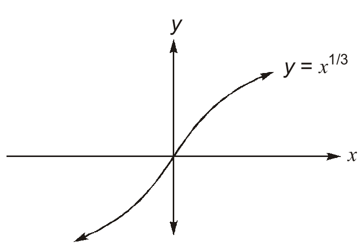


7. $y = kx^{1/2}$

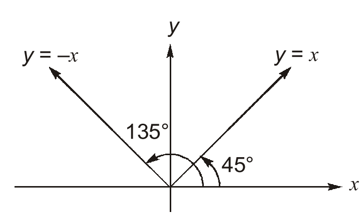
(similar shape for $kx^{1/4}$, $kx^{1/6}$...)



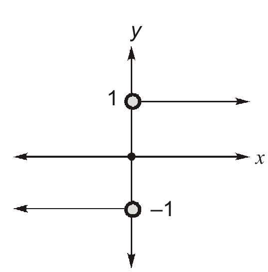
8. $y = kx^{1/3}$



9. $y = |x| = \begin{cases} -x & x < 0 \\ 0 & x = 0 \\ x & x > 0 \end{cases}$

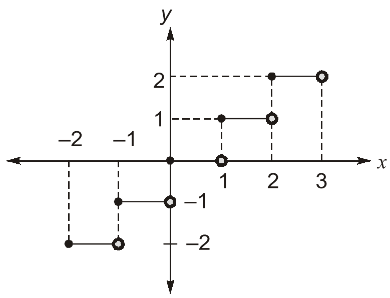


10. $y = \text{sgn}(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \rightarrow \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \\ 0 & \text{else} \end{cases}$



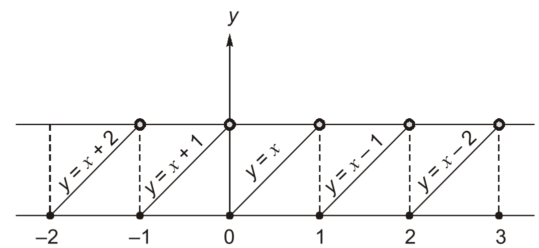
11. $y = [x]$, i.e., greatest integer function

$$y = [x] = \begin{cases} \vdots & \\ -2 & -2 \le x < -1 \\ -1 & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ \vdots & \end{cases}$$

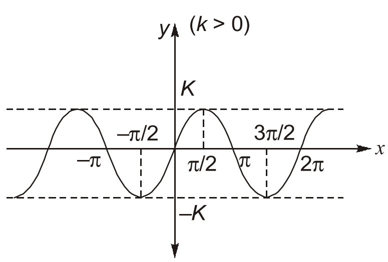


12. Fractional part function

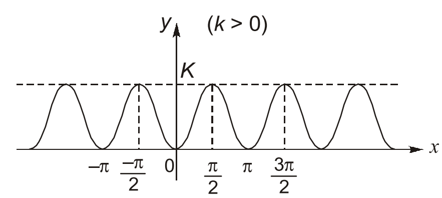
$$y = \{x\} = x - [x] = \begin{cases} \vdots & \\ (x+2) & -2 \le x < -1 \\ x+1 & -1 \le x < 0 \\ x & 0 \le x < 1 \\ \vdots & \end{cases}$$



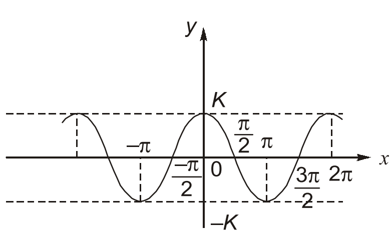
13. $y = K \sin x$



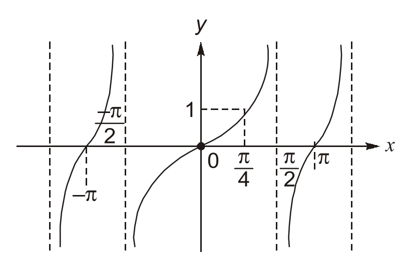
14. $y = K \sin^2 x$



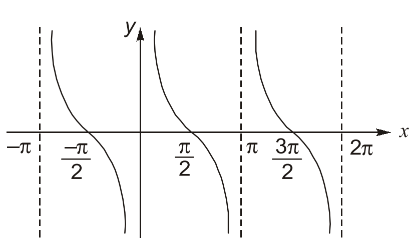
15. $y = K \cos x$



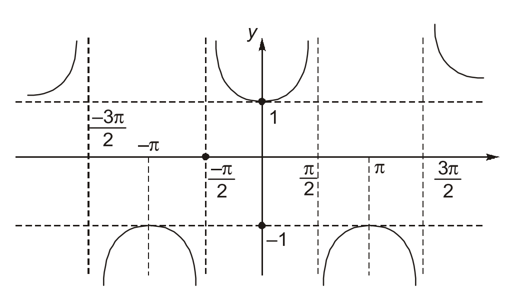
16. $y = \tan x$



17. $y = \cot x$



18. $y = \sec x$



Student's Assignments

Q.1 If $f(x) = \dfrac{2x^2 - 7x + 3}{5x^2 - 12x - 9}$, then $\lim_{x \to 3} f(x)$ will be

(a) $-\dfrac{1}{3}$ (b) $\dfrac{5}{18}$

(c) 0 (d) $\dfrac{2}{5}$

Q.2 $\lim_{\theta \to 0} \dfrac{\sin\left(\dfrac{\theta}{2}\right)}{\theta}$ is

(a) 0.5 (b) 1

(c) 2 (d) Not defined

Q.3 The Value of $\lim_{x \to 8} \dfrac{x^{1/3} - 2}{(x - 8)}$

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{12}$

(c) $\dfrac{1}{8}$ (d) $\dfrac{1}{4}$

Q.4 $\lim_{x \to \infty} \dfrac{x - \sin x}{x + \cos x}$ equals

(a) 1 (b) –1

(c) ∞ (d) $-\infty$

Q.5 The function $y = |2 - 3x|$

(a) is continuous $\forall x \in R$ and differentiable $\forall x \in R$

(b) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = \dfrac{3}{2}$

(c) is continuous $\forall x \in R$ and differentiable $\forall x \in R$ except at $x = \dfrac{2}{3}$

(d) is continuous $\forall x \in R$ except $x = 3$ and differentiable $\forall x \in R$

Q.6 What should be the value of λ such that the function defined below is continuous at $x = \dfrac{\pi}{2}$?

$$f(x) = \begin{cases} \dfrac{\lambda \cos x}{\dfrac{\pi}{2} - x} & \text{if } x \neq \dfrac{\pi}{2} \\ 1 & \text{if } x = \dfrac{\pi}{2} \end{cases}$$

(a) 0 (b) $\dfrac{2}{\pi}$

(c) 1 (d) $\dfrac{\pi}{2}$

Q.7 The expression $\lim_{\alpha \to 0} \dfrac{x^\alpha - 1}{\alpha}$ is equal to

(a) $\log x$ (b) 0

(c) $x \log x$ (d) ∞

Q.8 $\lim_{x \to 0} \left(\dfrac{e^{2x} - 1}{\sin(4x)}\right)$ is equal to

(a) 0 (b) 0.5

(c) 1 (d) 2

Q.9 The value of $\lim_{x \to 0} \dfrac{1 - \cos(x^2)}{2x^4}$ is

(a) 0 (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{4}$ (d) $\dfrac{1}{8}$

Q.10 The value of $\lim_{x \to \infty} (1 + x^2)^{e^{-x}}$ is

(a) 0 (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{2}$ (d) 1

Q.11 $\lim_{x \to \infty} \sqrt{x^2 + x - 1} - x$ is

(a) 0 (b) ∞

(c) $\dfrac{1}{2}$ (d) $-\infty$

Explanations

1. (b)

$$\lim_{x \to 3} f(x) = \underset{x \to 3}{\text{Lt}} \left(\frac{2x^2 - 7x + 3}{5x^2 - 12x - 9} \right)$$

Here this is of the form of $\left(\frac{0}{0}\right)$.

So, applying L-Hospital's rule

$$\lim_{x \to 3} \left(\frac{4x - 7}{10x - 12} \right) = \frac{5}{18}$$

2. (a)

$$\lim_{\theta \to 0} \frac{\frac{1}{2} \times \sin\left(\frac{\theta}{2}\right)}{\theta \times \frac{1}{2}} = \frac{1}{2} \lim_{\theta \to 0} \frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} = \frac{1}{2} = 0.5$$

3. (b)

$$(x - 8) = h \text{ (say)}$$

$$\Rightarrow \quad x = 8 + h$$

$$\therefore \quad \lim_{h \to 0} \frac{(8+h)^{1/3} - 2}{h}$$

Above form in the $\left(\frac{0}{0}\right)$ by putting the value $h = 0$

Applying L'Hospital rule

$$\lim_{h \to 0} \frac{\frac{1}{3}(8+h)^{\left(\frac{1}{3}-1\right)}}{1} = \frac{1}{3}(8)^{-\frac{2}{3}} = \frac{1}{12}$$

4. (a)

$$\lim_{x \to \infty} \frac{x - \sin x}{x + \cos x} = \lim_{x \to \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos x}{x}}$$

$$= \frac{\lim_{x \to \infty} \left(1 - \frac{\sin x}{x}\right)}{\lim_{x \to \infty} \left(1 + \frac{\cos x}{x}\right)}$$

$$= \frac{1 - \lim_{x \to \infty} \frac{\sin x}{x}}{1 + \lim_{x \to \infty} \frac{\cos x}{x}} = \frac{1 - 0}{1 + 0} = 1$$

5. (c)

$$y = |2 - 3x| = 2 - 3x \quad 2 - 3x \geq 0$$
$$= 3x - 2 \quad 2 - 3x < 0$$

Therefore, $y = 2 - 3x \quad x \leq \frac{2}{3}$

$$= 3x - 2 \quad x > \frac{2}{3}$$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only concern is at $x = \frac{2}{3}$

Left limit at $x = \frac{2}{3}$ is $2 - 3 \times \frac{2}{3} = 0$.

Right limit at $x = \frac{2}{3}$ is $3 \times \frac{2}{3} - 2 = 0$.

$$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$$

Since, Left limit = Right limit = $f\left(\frac{2}{3}\right)$,

Function is continuous at $\frac{2}{3}$.

y is therefore continuous $\forall\, x \in R$

Now since $2 - 3x$ and $3x - 2$ are polynomials, they are differentiable.

Only concern is at $x = \frac{2}{3}$.

Now, at $x = \frac{2}{3}$, LD = Left derivative = –3

RD = Right derivative = +3

LD ≠ RD

$\therefore$ The function y is not differentiable at $x = \frac{2}{3}$

So, we can say that y is differentiable $\forall x \in R$, except at $x = \frac{2}{3}$.

6. (c)

If $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \qquad \text{...(i)}$$